WHAT RESULTS ARE WE GETTING FROM GRAPHIC ALGEBRA?

By Arthur Whipple Smith.

I feel that I should explain to you that my acquaintance with the work of the secondary schools is entirely second hand and it may be that my ideas on the subject of graphics in secondary work are colored by what I may wish were possible instead of being entirely true to the facts. In my own experience as an instructor of freshmen I have found but little evidence indicating previous instruction in graphics and frequently the subject is thrust suddenly upon a freshman by the immediate needs of his college work. In many cases it is looked upon as only another novelty introduced to make college mathematics a thing to be dreaded and avoided if possible. I assume that it is proper for an instructor in first-year college work to expect a greater or less degree of familiarity with graphics on the part of incoming students and from this standpoint consider the question as to what may be gained by the study of graphics in connection with all branches of elementary mathematics. The subject should appeal to teachers of mathematics for at least three reasons, first, it is the simplest of our many symbolisms for magnitudes and in the order of nature precedes all the others: second, it often appeals to certain pupils who would otherwise be uninterested: third it affords connecting links among all branches of mathematics.

As to the first reason, Dr. Taylor, of Colgate University, has recently been giving some attention to the subject in connection with his work in the teachers' course and with his permission I shall quote from him.

"We have in general two methods of expressing our concepts and thoughts, viz: the graphic and the symbolic. The traveler makes known his adventures graphically by pictures and drawings as well as symbolically by words. To give clear and vivid ideas of magnitude the statistician represents them graphically by lines as well as symbolically by numbers. In the

order of nature the graphic method precedes the symbolic. The young child knows the picture horse before he knows or uses the word "horse." He often learns and can repeat illustrated poems when without the illustrations to interest and guide him he could not repeat a single word. The human race by the graphic method cultivated and perfected elementary geometry nearly two thousand years before the study of geometry by the symbolic method was dreamed of. When graphic geometry had been perfected the symbolic sciences of arithmetic and algebra were in their infancy. These facts would suggest that in our mathematical instruction to the young at least we should use the graphic method before we do the symbolic, but the reverse is our practice, the graphic being taught last, if at all. Pupils in arithmetic often do not suspect that there is any other *mathematical* way of expressing magnitudes or size than by numbers and when they are taught to represent distances and other magnitudes by the length of lines, *i. e.*, graphically, they generally fail to understand that the method is mathematical. Before a pupil begins the study of algebra, even while he is pursuing arithmetic, the graphic method of representing magnitudes should be made familiar as a method of mathematics. At this early stage the mathematical laboratory is the drawing room equipped with graded weights, spring balances, and other needed apparatus, where the pupil should learn how to represent distances, forces, and other magnitudes by lines or other drawings. For example, after he has decided how long a line he will use to represent the force needed to lift a weight of one pound, he should express by lines the forces needed to lift two pounds, three pounds, etc. He should also learn map drawing, not simply by copying the work of others but by making maps from given data, or better still, from data which he has himself collected. This graphical method would interest many to whom the symbolism of numbers is distasteful and at the same time might prove to be an excellent exercise in mathematics if properly done."

We thus see that the subject may quite properly be begun in the earliest of mathematical work and in a few particular instances I shall endeavor to point out concrete applications in geometry and algebra.

In plane geometry the usual theorems on areas, *e. g.*, those concerning the rectangles of sums and differences of lines, may be taken as the graphic forms of the algebraic theorems on the squares and products of sums and differences. Some writers on geometry introduce these algebraic theorems as corollaries to the geometric theorems.

In algebra the subject of surds usually proves a stumbling block to those who see no concrete meaning to such expressions as the square root of two, especially after he has been told that these are neither whole numbers or fractions. The pupil is quite likely to believe that such numbers are only the result of mathematical gymnastics. But here we find the advantage of graphics. We may recall the theorem on the square of the hypothenuse of a right triangle and form a figure by first constructing a right triangle with the legs each equal to one. On this triangle we build a second having as one leg the hypothenuse of the first triangle and the second leg equal to one. This process we continue indefinitely. It appears then that the lengths of the successive hypothenuses are to be represented symbolically, by the square roots of the successive integers 1, 2, 3, 4, 5, etc. The triangles are concrete, so is the hypothenuse of each, and the need of the symbols representing their lengths must be at once conceded.

In beginning algebra we encounter the difficulty of presenting to the beginner the idea of positive and negative numbers. That this is often poorly done is recognizable by the frequent statement on the part of pupils that all numbers in arithmetic are positive, thus failing to see that positive number implies the existence of negative number. Whenever I meet this idea in the mind of a pupil I ask him to define the meaning of the word *cold* if he were to assume that the entire universe was and always had been at a uniform temperature. In other words positive and negative are but names which we give to numbers which are used to denote the sizes of quantities which in addition to *size* possess the property of *oppositeness*, *i e.*, of mutual destruction when combined. Now a simple application of graphics may be used to fix these new ideas with clearness. We assume that the pupil has already learned in arithmetic to represent the relative sizes of magnitudes by both number symbols

and line segments. By introducing the subject of forces from familiar everyday illustrations we obtain the addition of direction to the lines used, including the idea of opposite directions. The next inquiry is how can both the directions and sizes of forces, which may be opposing forces, be represented by number symbols as well as by directed lines. The answer is that the size may be represented by the old arithmetic or size numbers, but that the idea of oppositeness requires the invention of *new* number symbols. Thus the need of numbers to represent both the size and the oppositeness of such forces introduces by definition two kinds of number, one called positive and the other negative, each possessing, in common with the other, *size* or *arithmetic value*. The rules for the addition and subtraction of such quality numbers are easily illustrated or even surmised from examples involving the composition of forces acting in either the same or opposite directions. A further help in this direction and one which has important uses later is the scheme of geographic latitude and longitude, in which the notations N and S (or E and W), are replaced by the use of positive and negative numbers respectively.

It is impossible to give any concrete meaning to a single equation in two unknowns if we limit the treatment to the question of its solution but it is possible to do so by means of its graphic representation, in the discussion of which we find geometric meanings not only for the equations individually but as well for all the possible peculiarities which may arise in discussing them simultaneously. The simplest introduction to this graphical study is a review of geographical latitude and longitude. Noting here the oppositeness as expressed by N and S, E and W, the pupil thus learns that any point on the earth's surface (or on a plane, if we draw two perpendicular lines to replace the equator and principal meridian), can be represented by two *quality-numbers* denoting the distances and directions of the given point from the two fixed reference lines. The converse—that every two such numbers define the position of a point—easily follows.

If now we recall the locus idea of geometry, *i. e.*, that the locus is an aggregate of points bearing some definite relation to each other or to some given configuration, it is easily inferred

that the pairs of numbers which represent these points must bear some definite relation to each other. If now we lead the pupil to see that there is an unlimited number of solutions, *i. e.*, pairs of numbers, for each linear equation in two unknowns and recall to him that each of these pairs represents, or is pictured by, a point we have opened the way to the graphics of the equation. It is evident that these many pairs of numbers cannot be chosen at random for each such pair must satisfy the equation and hence the points which picture these pairs cannot be chosen at random but are subject to a condition, *i. e.*, they form a locus of some sort. There are then two ways of expressing a locus, either by an equation (*i. e.*, symbolically) or by a graph. If these conclusions be granted we have the following simple theorems:

Every equation in two unknowns represents a relation connecting pairs of numbers (its solutions). The graphs of such related pairs of numbers form a geometric locus.

By a solution of two equations in two unknowns we mean that pair (or pairs) of numbers which is common to both, and hence geometrically such a common solution must represent the point (or points) which are common to the two loci corresponding to the equations, *i. e.*, their intersections.

If two equations have no common solution, their loci do not intersect. If two equations have a finite number of common solutions, their loci have the same number of points of intersection. If two equations are equivalent, their loci coincide. A little thought will show the graphic meaning for every condition which may arise in the algebraic solution of the equations.

I do not think it will prove too difficult for those of the class who show any curiosity at all as to the character of these loci to understand that the linear equation in two unknowns represents a straight line. The following proof requires only the most elementary knowledge of the equation and a little geometry.

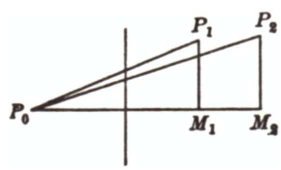
We start with the general equation $ax+by=c$ and solve it for y, obtaining

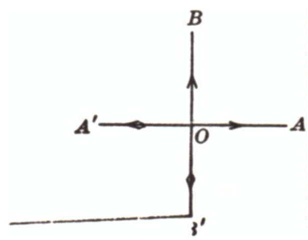
$$y=\frac{c-ax}{b}.$$

For simplicity we represent the fractions a/b and c/a by

the letters $-m$ and d. The equation now takes the form $y = m(x - d)$. One solution of this equation is (d, o) the picture of which we call P_0. We suppose that x_1y_1 and x_2y_2 are two other solutions the pictures of which we represent by P_1 and P_2.

We now join the points P_0 and P_1 and P_0 and P_2 by straight lines forming with the perpendiculars M_1P_1 and M_2P_2 two right triangles.





From the figure we find

$$\begin{aligned} P_0M_1 &= x_1 - d, \\ P_0M_2 &= x_2 - d, \\ M_1P_1 &= y_1, \\ M_2P_2 &= y_2, \end{aligned}$$

and by using the given equation we obtain

$$\frac{M_1P_1}{P_0M_1} = m = \frac{M_2P_2}{P_0M_2}.$$

Hence the two right triangles are similar and angles $M_1P_0P_1$ and $M_2P_0P_2$ are equal, *i. e.*, the straight line joining the two points of the locus P_0 and P_1 passes through any third point P_2. The locus is therefore a straight line.

Some may say that pupils in secondary work are not prepared for such reasoning. My own opinion is that such work is preferable to spherical trigonometry and the study of series which subjects may freshmen claim to have had.

And furthermore the future study of the equation will naturally bring to the minds of most pupils a desire to know more of the graphical side, *i. e.*, of analytical geometry, a result which is certainly much to be desired. If the teacher can point out to them as facts (the proof of which they can learn later)

that the heavenly bodies which move about the sun always travel in one of two kinds of orbits, viz: ellipses or parabolas, that these curves are called conic sections because they may be cut from the right circular cone, that they are to be studied later in connection with the study of equations of the second degree in two unknowns, I feel certain that the algebraic equation will at once become a thing of importance in the eyes of most pupils.

For those who teach the subject of imaginary and complex number there is no better way of approach than the graphic. We consider the problem of forces at right angles to each other. The use of positive and negative numbers is of course sufficient for representing the opposite forces in either of the lines of direction but the problem here calls for numbers of two kinds which cannot by ordinary processes of addition and subtraction be combined into a single number. The definition of such new numbers may be made as follows. Let equal forces at right angles to each other be represented by the four lines OA, OA', OB, OB'.

Let the magnitude of these equal forces be represented by the *arithmetic* number a. We then designate OA by the *algebraic* number ^{+}a and OA' by ^{-}a. According to the definition of positive and negative numbers these may be written as $^{+}1 \cdot a$ and $^{-}1 \cdot a$ respectively, a notation which suggests the possibility of a similar one for representing OB and OB'. We then represent OB and OB' by the symbols $^{+}i \cdot a$ and $^{-}i \cdot a$ respectively. As an arbitrary notation this might be sufficient but we desire also to find some relation between these two kinds of number symbols.

The vector OA becomes OB graphically by a counterclockwise revolution of 90° and symbolically by using the factor i. Also OA becomes OA' by a similar revolution of 180° and symbolically by using the factor -1. If we attempt to identify this graphic process with the symbolic we are forced to set the 180° in correspondence to the factor -1 and 90° in correspondence to the factor i. But since 180° is formed by two consecutive 90° revolutions, the factor -1 should be identical with the successive applications of i and i, *i. e.*, of i^2. Thus we obtain $-1 \equiv i^2$ or $i \equiv i\sqrt{-1}$, a number scheme which will be found to be quite self-consistent and from which the theory of imaginaries is easily derived.

The few examples of graphics which I have given are, I think, sufficient to point out that this method cannot help but weld together subjects which in the minds of too many are quite distinct. I often wonder if many of those who so dislike the subject in general would not change their opinion if they were to keep at it until they reach analytics, for example. Algebra is so often taught as mere machinery for grinding out answers with grades accordingly, that I do not wonder at the dislike for the subject. We must seek to make mathematics interesting without losing any of its rigorousness and one of the simplest ways is to knit all its branches together early in the course and keep before the pupil, so far as possible, a hint as to what he is approaching. For the teacher I know no easier way than the graphical one.

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There is an idea abroad among people that they should make their neighbors good. One person I have to make good: myself. But my duty to my neighbor is much more nearly expressed by saying that I have to make him happy—if I may.

A man who has a few friends or one who has a dozen (if there be any one so wealthy on this earth), cannot forget on how precarious a basis his happiness reposes; and how by a stroke or two of fate—a death, a few light words, a piece of stamped paper, or a woman's bright eyes—he may be left in a month destitute of all.—*Robert Louis Stevenson.*